

Anisotropic metamaterial as an analogue of a black hole

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Abstract

Propagation of light in a metamaterial medium which mimics curved spacetime and acts like a black hole is studied. We show that for a particular type of spacetimes and wave polarization, the time dilation appears as dielectric permittivity, while the spatial curvature manifests as magnetic permeability. The optical analogue to the relativistic Hamiltonian which determines the ray paths (null geodesics) in the anisotropic metamaterial is obtained. By applying the formalism to the Schwarzschild metric, we compare the ray paths with full-wave simulations in the equivalent optical medium.

Keywords: Metamaterials, Anisotropic optical materials, Analogue gravity, Black hole, Schwarzschild spacetime

1. Introduction

One of the hot topics of modern technology is to build artificial materials whose permittivity and permeability can be properly engineered by incorporating structural elements of subwavelength sizes. As a result, one can create materials (called *metamaterials*) with the desired electromagnetic response which offers new opportunities for realizing such exotic phenomena as negative refraction, cloaking, super-lenses for subwavelength imaging, microantennas, etc. [1, 2].

Recently, it has also been recognized that metamaterials can be used to mimic general-relativity phenomena [3, 4]. The propagation of electromagnetic waves in curved spacetime is formally equivalent to the propagation in flat spacetime in a certain inhomogeneous anisotropic or bianisotropic medium [5–12]. Based on this equivalence, different general relativity phenomena have been discussed from the point of view of possible realization in metamaterials: optical analogues of black holes [13–18], Schwarzschild spacetime [12, 19], de Sitter spacetime [20–22], cosmic strings [23, 24], wormholes [25], Hawking radiation [26], the “Big Bang” and cosmological inflation [27, 28], colliding gravitational waves [29], among others.

The deflection of light waves in gradient-index optical materials mimicking optical black holes was studied theoretically [13, 14, 18] and experimentally [15–18]. These materials, called by authors “omnidirectional electromagnetic

absorbers”, are characterized by an isotropic effective refractive index. A real cosmological black hole (BH) can often be described by an anisotropic spacetime, as, for example, the case of the Schwarzschild BH. For that case, one should determine the permittivity and permeability tensors instead of the refractive index in order to introduce the equivalent optical medium [12, 19]. Chen et al. [19] simulated the wave propagation outside the Schwarzschild BH and observed in their numerical results the phenomenon of “photon sphere”, which is an important feature of the BH system. It would be interesting to go further and study the propagation of light waves in optically anisotropic media which mimic cosmological BHs and compare the results with ray paths obtained from the Hamiltonian method.

The aim of this letter is twofold. First, we determine the constitutive relations of an inhomogeneous anisotropic medium which is formally equivalent to the static spacetime metric obeying rotational symmetries and can be applied, in principle, to the medium either in isotropic or anisotropic form. Second, by making use of the eikonal approximation to the wave equation, we obtain the expression for the optical Hamiltonian which we found to be identical to the one obtained from general relativity for null geodesics, but different from the optical Hamiltonian used in Refs. [13, 15, 30, 31]. Then we apply the formalism to the Schwarzschild spacetime that is a solution to the Einstein field equations in vacuum [32]. We compare the wave propagation with the ray dynamics outside the BH in the effective medium and obtain a very good correspondence. As an interesting feature we find that light does not propagate in the direction of the wave normal, there is an angle between the wave velocity and the ray velocity. The obtained results are discussed from the point of view of metamaterial implementation.

2. General relativity in a metamaterial medium

2.1. Medium parameters

Long time ago, Tamm pointed out the parallels between anisotropic crystals and curved spacetimes [5]. Later studies showed [6–11] that the propagation of light in empty curved space distorted by a gravitational field is formally equivalent to light propagation in flat space filled with an inhomogeneous anisotropic medium.

Indeed, consider a spacetime background with a general metric ¹

$$ds^2 = g_{00} dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j, \quad (1)$$

where $i, j = 1, 2, 3$ run over arbitrary spatial coordinates. Then, it can be shown [11] that the covariant Maxwell’s equations written in curved coordinates can be transformed into their standard form for flat space but in the presence of

¹From now on we follow the standard notations for covariant (subindices) and contravariant (superindices) quantities.

an effective medium. The constitutive relations of the equivalent medium have been found in the form [7]:

$$D^i = \varepsilon^{ij} E_j - (\mathbf{\Gamma} \times \mathbf{H})^i, \quad B^i = \mu^{ij} H_j + (\mathbf{\Gamma} \times \mathbf{E})^i, \quad (2)$$

which connect the fields \mathbf{D} , \mathbf{B} , \mathbf{E} and \mathbf{H} via nontrivial permittivity and permeability tensors

$$\varepsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g}}{g_{00}} g^{ij} \quad (3)$$

and a vector $\mathbf{\Gamma}$ given by

$$\Gamma_i = -\frac{g_{0i}}{g_{00}}. \quad (4)$$

Here, g^{ij} is the inverse of g_{ij} and g is the determinant of the full spacetime metric $g_{\mu\nu}$, with $\mu, \nu = 0, 1, 2, 3$. Note that the information about the gravitational field is essentially embedded in the material properties of the effective medium: the tensors ε^{ij} , μ^{ij} which are symmetric and should be equal, and the vector $\mathbf{\Gamma}$ which couples the electric and magnetic fields. The invention of metamaterials during the last decade [1, 2] opened up the possibility to design electromagnetic media corresponding to different spacetimes [3, 4, 19–29].

In this letter, we consider a static spacetime metric associated with a spherically symmetric cosmological BH. Due to time-reversal symmetry, $g_{0i} = 0$ and the coupling between the electric and magnetic fields vanishes, $\mathbf{\Gamma} = 0$. The metric (1) in (t, r, θ, φ) coordinates can then be written in a generic form as [32]

$$ds^2 = g_{00}(r) dt^2 + g_{rr}(r) \{dr^2 + f(r) [r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2]\}, \quad (5)$$

where f is the “anisotropic factor”. Note that the metric (5) obeys the rotational symmetries in the three-dimensional (r, θ, φ) space.

Then, we have to project the metric (5) into a flat background to obtain the medium parameters in the Cartesian coordinate system. To do that, we apply a coordinate transformation and, from Eq. (3), we get the permittivity and permeability tensors in the form:

$$\varepsilon^{ij} = \mu^{ij} = \sqrt{-\frac{g_{rr}}{g_{00}}} \left[\delta^{ij} - (1-f) \frac{x^i x^j}{r^2} \right], \quad (6)$$

where δ^{ij} is the Kronecker delta, $r = \sqrt{x^2 + y^2 + z^2}$, and we denoted the Cartesian coordinates $(x^1, x^2, x^3) \equiv (x, y, z)$. It is seen that whenever $f \neq 1$, the permittivity and permeability tensors contain the off-diagonal elements and the equivalent medium is essentially anisotropic. Only in the case of $f = 1$, the off-diagonal elements vanish and the medium becomes completely isotropic with all the diagonal elements equal to the refractive index: $n(r) = \sqrt{-g_{rr}/g_{00}}$. Note that in general relativity the spacetime with $f = 1$ in Eq. (5) is said to be conformal to flat space. Every static spherically symmetric spacetime with $f \neq 1$ can be converted to conformally flat form by an appropriate transformation of

the radial coordinate: $r \rightarrow \rho$. The new radial coordinate is obtained by

$$\rho = r \exp \left\{ \int_r^\infty \left[1 - \frac{1}{\sqrt{f(r')}} \right] \frac{dr'}{r'} \right\}, \quad (7)$$

where the isotropic boundary condition at infinity, $f(\infty) = 1$, is taken into account. The line element in the $(t, \rho, \theta, \varphi)$ *isotropic* coordinates takes the conformally flat form:

$$ds^2 = g_{00}[r(\rho)] dt^2 + \Lambda(\rho) (d\rho^2 + \rho^2 \sin^2 \theta d\varphi^2 + \rho^2 d\theta^2), \quad (8)$$

where the time dilation term g_{00} and the conformal factor $\Lambda = g_{rr} f r^2 / \rho^2$ are calculated by means of the function $r(\rho)$ which should be obtained by inverting (7). Thus, the permittivity and permeability tensors are simply reduced to the isotropic refractive index:

$$\varepsilon^{ij} = \mu^{ij} = \delta^{ij} \sqrt{-\frac{\Lambda}{g_{00}}} \equiv \delta^{ij} n(\rho). \quad (9)$$

The equivalent medium determined by Eq. (9) is still inhomogeneous since the refractive index varies with the radial coordinate, but the light velocity in the medium becomes isotropic, a property that is much simpler to implement in metamaterial design.

In what follows, we will compare the results for light propagation in two different equivalent media – isotropic and anisotropic – both corresponding to the same spacetime metric in order to see the physical differences.

2.2. Electromagnetic fields. TE and TM polarizations

The results from the previous section indicate that an electromagnetic field can be thought of as propagating in flat background but in the presence of a medium whose properties are constructed from a curved spacetime. The fields, for the static case we consider, are related by:

$$D^i = \varepsilon^{ij} E_j, \quad B^i = \mu^{ij} H_j. \quad (10)$$

Due to the anisotropy of the medium for the metric in nonconformally flat form, the electric displacement field \mathbf{D} is not in the direction of \mathbf{E} , and the magnetic induction field \mathbf{B} is not in the direction of \mathbf{H} . To simplify the treatment of the problem, we consider the propagation of light in the equatorial plane, $z = 0$. In such a case, one of the anisotropies – electric or magnetic – can be eliminated.

Indeed, consider the TE polarization for an electromagnetic wave for which \mathbf{E} is perpendicular to the x - y plane. Equation (6) for $z = 0$ leads to $\varepsilon^{xz} = \varepsilon^{yz} = 0$, hence the directions of \mathbf{D} and \mathbf{E} coincide. This means that ε^{zz} is the only relevant matrix element which connects the nonzero electric components of the field: $D^z = \varepsilon^{zz} E_z$, and the electric anisotropy of the medium is irrelevant. As for the magnetic components, we obtain $\mu^{xz} = \mu^{yz} = 0$, $B^z = H^z = 0$, and

$B^x = \mu^{xx}H_x + \mu^{xy}H_y$ and $B^y = \mu^{xy}H_x + \mu^{yy}H_y$. Hence, \mathbf{B} and \mathbf{H} are confined to the x - y plane, but their directions do not coincide.

Similarly, one can consider the TM polarization for which \mathbf{H} is perpendicular to the x - y plane. In such a case, one gets $B^z = \mu^{zz}H_z$, $D^x = \varepsilon^{xx}E_x + \varepsilon^{xy}E_y$, and $D^y = \varepsilon^{xy}E_x + \varepsilon^{yy}E_y$. This means that the directions of \mathbf{B} and \mathbf{H} coincide, and the magnetic anisotropy is irrelevant. It should be noted that, since $\varepsilon^{ij} = \mu^{ij}$ and they are both symmetrical, the TE and TM polarizations are completely equivalent. These two cases are summarized in Fig.1.

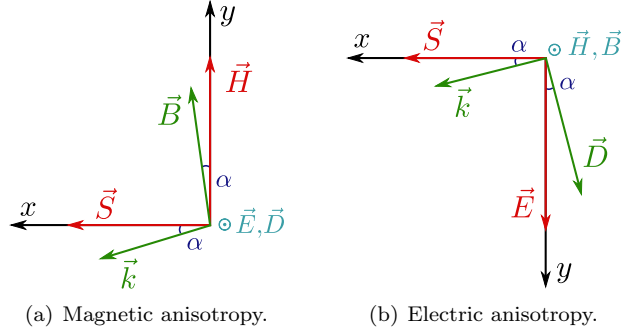


Figure 1: Electromagnetic fields \mathbf{H} , \mathbf{B} , \mathbf{E} , \mathbf{D} , wavevector \mathbf{k} and Poynting vector \mathbf{S} for: (a) TE polarization in a magnetically anisotropic medium, and (b) TM polarization in an electrically anisotropic medium.

The medium anisotropy has very important physical consequences on the propagation of the electromagnetic waves. If we look at the monochromatic plane wave with wavevector \mathbf{k} , then \mathbf{D} , \mathbf{B} and \mathbf{k} form an orthogonal vector triplet. On the other hand, \mathbf{E} , \mathbf{H} and the Poynting vector \mathbf{S} form another orthogonal vector triplet. Therefore, the energy does not propagate in the direction of the wave normal [33] and the ray velocity and the wave velocity are not equal, unlike what happens for an isotropic medium. It is seen that the angle between \mathbf{S} and \mathbf{k} (denoted by α in Fig.1) is equal to that between \mathbf{B} and \mathbf{H} in the TE polarization or between \mathbf{D} and \mathbf{E} in the TM polarization. For the angle α we find:

$$\tan \alpha = -\frac{\mu^{xy}}{\mu^{yy}} = -\frac{\varepsilon^{xy}}{\varepsilon^{yy}} = \frac{(1-f)xy}{x^2 + fy^2} \quad (11)$$

Only when $f = 1$, the medium becomes isotropic, α vanishes and the ray and wave velocities become equal.

2.3. Schwarzschild spacetime

Equations (6) and (9) can now be applied to any static spherically symmetric spacetime. As an example, we consider the Schwarzschild metric that is a solution to the Einstein field equations in vacuum [32]. This metric in spherical

coordinates reads:

$$ds^2 = - \left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2(\sin^2 \theta d\varphi^2 + d\theta^2). \quad (12)$$

Here, the speed of light $c = 1$, the Schwarzschild radius $r_s = 2GM/c^2$, G is the universal gravitational constant and M is the mass of the source of gravitation. Comparing with Eq. (5), one can easily identify: $g_{00} = -(1 - r_s/r)$, $g_{rr} = -1/g_{00}$, and $1 - f = r_s/r$. Since the Schwarzschild metric (12) is not in a conformally flat form, one obtains the anisotropic equivalent medium tensors

$$\varepsilon^{ij} = \mu^{ij} = \frac{1}{1 - \frac{r_s}{r}} \left(\delta^{ij} - \frac{x^i x^j}{r^3} r_s \right), \quad (13)$$

and from Eq. (11) we find the angle between the wave and ray velocities:

$$\tan \alpha = \frac{xyr_s}{r^3 - y^2 r_s} \quad (14)$$

Equation (13) is a compact version of the medium tensors which can also be found as matrices in Refs. [19, 22].

The Schwarzschild metric can be written in a conformally flat form by means of the well known substitution (see, e.g. [32])

$$r = \rho \left(1 + \frac{r_s}{4\rho} \right)^2. \quad (15)$$

The inverted formula for the isotropic radial coordinate ρ reads:

$$\rho = \frac{1}{2} \left(r - \frac{1}{2} r_s + \sqrt{r^2 - r_s r} \right) \quad (16)$$

which can also be obtained from Eq. (7) by substituting $f = 1 - r_s/r$. In isotropic coordinates, the Schwarzschild metric becomes [32]

$$ds^2 = - \left(\frac{1 - \frac{r_s}{4\rho}}{1 + \frac{r_s}{4\rho}} \right)^2 dt^2 + \left(1 + \frac{r_s}{4\rho} \right)^4 (d\rho^2 + \rho^2 \sin^2 \theta d\varphi^2 + \rho^2 d\theta^2). \quad (17)$$

Finally, the refractive index of the equivalent medium is found from Eq. (9) as (see also Ref. [9])

$$n(\rho) = \frac{\left(1 + \frac{r_s}{4\rho} \right)^3}{1 - \frac{r_s}{4\rho}}, \quad (18)$$

The formulas given in this section for the Schwarzschild spacetime will be used in Sec. 5 for numerical simulation.

3. Maxwell's equations in anisotropic medium

The starting point of our analysis of wave propagation in the equivalent medium is the Maxwell equations written in a source-free form

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \quad (19)$$

These equations should be supplemented by the constitutive relations (10) with the medium tensors (6) expressed through the spacetime metric coefficients. If the operating frequency bandwidth is sufficiently narrow, the dispersion can be neglected and one can consider a monochromatic wave with frequency ω . Equations (19) can then be reduced [34] to the wave equation for a time-harmonic electric field vector \mathbf{E}_ω

$$\nabla \times \left[\underline{\underline{\mu}}^{-1} (\nabla \times \mathbf{E}_\omega) \right] = \omega^2 \underline{\underline{\varepsilon}} \mathbf{E}_\omega. \quad (20)$$

The medium is generally anisotropic, hence, $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\mu}}^{-1}$ denote the permittivity and inverse permeability tensors, respectively.

As it was discussed in Sec. 2.2, the electromagnetic propagation in the equatorial plane, $z = 0$, can be described in terms of either TE or TM waves. For the TE wave, $\mathbf{E}_\omega(x, y, z) = E(x, y)e^{-ik_z z}\hat{\mathbf{z}}$, we obtain the equation for the z -component of the electric field E in the two-dimensional (x, y) space

$$\frac{\partial}{\partial x} \left(\mu_{xy} \frac{\partial E}{\partial y} - \mu_{yy} \frac{\partial E}{\partial x} \right) - \frac{\partial}{\partial y} \left(\mu_{xx} \frac{\partial E}{\partial y} - \mu_{xy} \frac{\partial E}{\partial x} \right) = \omega^2 \varepsilon^{zz} E, \quad (21)$$

where $\mu_{xx}, \mu_{xy}, \mu_{yy}$ are the corresponding components of the tensor $\underline{\underline{\mu}}^{-1}$ and ε^{zz} corresponds to $\underline{\underline{\varepsilon}}$. Since only one component of $\underline{\underline{\varepsilon}}$ has entered into the final equation (21), the permittivity can be chosen isotropic with all its diagonal elements equal to ε^{zz} and the off-diagonal terms equal to zero. In this case, the equivalent medium will be magnetically anisotropic [see Fig. 1(a)].

The wave equation for the TM case can also be easily obtained from the above formulas by means of the substitutions: $\varepsilon^{ij} \rightleftharpoons \mu^{ij}$, $E \rightarrow H$. In such a case, the medium is electrically anisotropic [Fig. 1(b)], and one should solve the equation for the magnetic field $H(x, y)$.

We would like to point out that we do not use the plane wave approximation to treat the problem, instead, we will solve the wave equation (21) numerically for a Gaussian beam with the appropriate boundary conditions. This situation, as we believe, is more realistic from the experimental point of view.

4. Hamiltonian formulation

In the limit of geometrical optics, for which the wavelength is much smaller than the scale of variation of the medium parameters, electromagnetic waves should follow ray trajectories (null geodesics of the background spacetime). To test the validity of our scheme, it would be interesting to compare the full-wave

numerical calculation of the propagation of light with its geometrical optics limit, for which analytical solutions exist. To accomplish this task, we have to define the Hamiltonian.

Let us start with the wave equation (20), which we apply again to the equatorial plane, $z = 0$, and consider the TE polarization for the wave propagation. Given that the medium tensors are diagonal in the spherical coordinate system, it is advantageous to rewrite Eq. (21) in polar coordinates: $E(x, y) \rightarrow E(r, \varphi)$. One gets,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\mu_\varphi} \frac{\partial E}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left(\frac{1}{\mu_r} \frac{\partial E}{\partial \varphi} \right) + \omega^2 \varepsilon_z E = 0. \quad (22)$$

where we denote $\mu_r \equiv \mu^{rr}$, $\mu_\varphi \equiv \mu^{\varphi\varphi}$, and $\varepsilon_z \equiv \varepsilon^{zz}$, which are the medium parameters in polar coordinates.

In the eikonal approximation we seek the solution in the form: $E(r, \varphi) = E_0 \exp[iS(r, \varphi)]$, where E_0 is the slowly varying amplitude and $S(r, \varphi)$ is the fast varying eikonal [35]. Substituting into Eq. (22) and neglecting the small terms corresponding to the derivatives over slowly varying functions, we obtain:

$$\frac{1}{\mu_\varphi} \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{\mu_r r^2} \left(\frac{\partial S}{\partial \varphi} \right)^2 - \varepsilon_z \omega^2 = 0, \quad (23)$$

which can be rewritten in terms of the spacetime metric coefficients as

$$\frac{1}{g_{rr}} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{f r^2} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right] + \frac{1}{g_{00}} \omega^2 = 0. \quad (24)$$

But this is nothing else than the eikonal equation for a light ray in a gravitational field [35]

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} = 0 \quad (25)$$

with the indices α, β running over space and time. The Hamiltonian representation of Eq. (25) is obtained by rewriting it in terms of the variables $p_\alpha = \partial S / \partial x^\alpha$, where p_α is the 4-momentum. One gets

$$\mathcal{H} = g^{\alpha\beta} p_\alpha p_\beta = p^\beta p_\beta = 0. \quad (26)$$

Thus, with the definitions $p_r = \partial S / \partial r$, $p_\varphi = \partial S / \partial \varphi$, and $p_t = \omega$, just plugging them into Eq. (24), we obtain

$$\mathcal{H} = \frac{1}{g_{00}} p_t^2 + \frac{1}{g_{rr}} \left(p_r^2 + \frac{1}{f r^2} p_\varphi^2 \right). \quad (27)$$

The optic Hamiltonian \mathcal{H} given by Eq. (27) describes the ray trajectories in the equivalent anisotropic medium and is equivalent to the one used to determine the null geodesics in the gravitational field. It takes into account the time dilation by means of the term g_{00} , the spatial curvature by the metric coefficient g_{rr} and the medium anisotropy through the function f . The latter should result in the deviation of the momentum vector \mathbf{p} from the ray propagation direction.

Having found the Hamiltonian \mathcal{H} , the ray dynamics can easily be calculated from Hamilton's equations of motion:

$$\dot{q}^\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q^\alpha}, \quad (28)$$

where the canonical coordinates in our case are: $q^\alpha = (t, r, \varphi)$, $p_\alpha = (p_t, p_r, p_\varphi)$ and the “dot” denotes the derivative over the affine parameter which varies along the trajectory. Note that in the Hamiltonian (27), the coefficients g_{00} , g_{rr} and f are functions of only the radial coordinate. Therefore, there exist two integrals of motion, $\dot{p}_t = 0$ and $\dot{p}_\varphi = 0$, which just state the conservation of energy and angular momentum for the static spherically symmetric system. In principle, these two constants of motion can be introduced; but in fact, since photons are massless, the ray trajectories are determined by only one constant – the impact parameter b which is the ratio between them [36].

We can now determine the ray trajectories for a particular spacetime metric by means of Eqs. (28) with the appropriate boundary conditions. To be specific, we consider the Schwarzschild metric (12) for which the Hamiltonian can be readily obtained by means of Eq. (27)

$$\mathcal{H} = -\frac{1}{1 - \frac{r_s}{r}} p_t^2 + \left(1 - \frac{r_s}{r}\right) p_r^2 + \frac{1}{r^2} p_\varphi^2. \quad (29)$$

On the other hand, in isotropic coordinates (see Eq. (17)), the Schwarzschild Hamiltonian becomes

$$\mathcal{H} = -\left(\frac{1 + \frac{r_s}{4\rho}}{1 - \frac{r_s}{4\rho}}\right)^2 p_t^2 + \frac{1}{\left(1 + \frac{r_s}{4\rho}\right)^4} \left(p_\rho^2 + \frac{1}{\rho^2} p_\varphi^2\right). \quad (30)$$

For both cases, isotropic and anisotropic, we will compare different kinds of trajectories with the full-wave numerical simulations in Sec. 5. Note that Eq. (26), when applied to the optical medium, gives the generalized dispersion relation [4] also known as the Fresnel equation [33, 34]. The discussion of this relation for the Schwarzschild equivalent medium can be found, e.g., in Ref. [12].

It would be appropriate to make some important remarks here.

- (i) By comparing the terms in Eqs. (23) and (24), one can see that for the TE wave, the time dilation term g_{00} is determined by the dielectric permittivity of the equivalent medium, while the spatial metric components g_{rr} and f depend on the magnetic permeability. In other words, the time and space metric coefficients correspond to different properties of the medium: either electric or magnetic. Similarly, if we consider the TM wave, one should replace: $\varepsilon^{ij} \rightleftharpoons \mu^{ij}$; $E \rightarrow H$ (see Sec. 2.2), and the correspondence will be just the opposite: g_{00} is related to μ_z , while g_{rr} and f correspond to ε_r and ε_φ .
- (ii) Equation (23) can be divided by ε_z and the medium parameters can be re-defined according to: $\tilde{\mu}_\varphi = \mu_\varphi \varepsilon_z$, $\tilde{\mu}_r = \mu_r \varepsilon_z$ and $\tilde{\varepsilon} = 1$. It is easy to check

that the new medium gives exactly the same ray trajectories, although it has the permittivity $\tilde{\epsilon}$ of free space and the inhomogeneous permeability $\tilde{\mu}$. Actually, the Hamiltonian \mathcal{H} may be multiplied by any arbitrary function of the coordinates (by g_{00} , in this case) without changing the ray path obtained from the equations of motion since only the parametrization of the affine parameter changes. This means that in the geometrical optics limit, one can introduce a set of equivalent media, all giving the same light ray paths. But the wave equation (22) (or in general Eq. (21)) cannot be renormalized in this way, and therefore the interference patterns for those analogous media will be different.

5. Comparison of wave propagation with ray dynamics

In this section we present the results obtained numerically for the propagation of light waves in an anisotropic medium which mimics a static spherically symmetric cosmological spacetime. As an example, we consider the Schwarzschild spacetime with the goal to compare the results for two sets of coordinates: spherical and isotropic. On the other hand, to validate our theoretical approach, we wish to compare the full-wave numerical simulation with the ray trajectories obtained within the Hamiltonian framework.

We solve Eq. (21) in a rectangular 2D geometry of (x, y) space for a TE polarized wave injected from the right (see Figs. 2 - 3). The medium parameters we use are given by Eq. (13) for the anisotropic spacetime and by Eq. (18) for the isotropic spacetime. In both cases, we solve the equations in Cartesian coordinates. The computational domain is surrounded by a perfectly matched layer that absorbs the outward waves to ensure that there are no unwanted reflections, and the simulations are done by means of a standard software solver.

It is known that the Schwarzschild black hole presents an event horizon at $r = r_s$ (in isotropic coordinates, at $\rho = r_s/4 \equiv \rho_s$) where the gravity is so strong, that light cannot escape [36]. The metric has a coordinate singularity there, which translates into singular medium parameters. To avoid it, we set an effective horizon at $r_s + \delta$ with δ being small positive number² and impose an absorbing inner core for $r < r_s + \delta$ [13, 19].

We use a Gaussian shape for the TE wave injected at the boundary, however, care should be taken for the injection direction. If the medium is anisotropic, the beam does not propagate in the direction of the wave normal, as we discussed in Sec. 2.2. There will be an angle α between the wave velocity and the ray velocity (see Fig. 1). To ensure that the wave is always injected in a controlled way, in our case along the x -axis, independently of the point of injection (that is, for different impact parameters b), we impose: $\mathbf{n} = -(\cos \alpha)\hat{\mathbf{x}} - (\sin \alpha)\hat{\mathbf{y}}$ for the normal unit vector \mathbf{n} of the wave front. The angle α is determined by Eq.

²We set $\delta = 0.05 r_s$ (anisotropic case) and $\delta = 0.11 r_s$ (isotropic) in the calculations. We have checked that the variation of δ , whenever it is small enough, does not affect the global distribution of fields in the whole domain except some tiny shell at the horizon.

(11) applied to the point of injection (x_0, y_0) . The fields correspond to those depicted in Fig. 1(a), in particular, the electric field vector is perpendicular to the plane of simulation and the medium has a magnetic anisotropy. Figure 1(b) corresponds to the case of TM wave injection into an electrically anisotropic medium. This case, in principle, can also be simulated. Because the injected wave is not a plane wave, the wavevector \mathbf{k} in Fig. 1 should be replaced by the wave normal vector \mathbf{n} of the Gaussian beam.

The impact parameter b is the key quantity which allows us to distinguish between different types of ray trajectories [36]. Depending on its value, one may observe capture ($b < b_c$) or deflection ($b > b_c$). Its critical value $b_c = 3\sqrt{3}r_s/2$ determines the so-called photonic sphere, which corresponds to an unstable circular orbit of radius $3r_s/2$. To relate the value of b with the initial coordinates (x_0, y_0) , we use the following formula

$$b = y_0 \sqrt{-\frac{g_{rr}}{g_{00}}} \frac{r_0 f}{\sqrt{x_0^2 + f y_0^2}} \quad (31)$$

with the metric parameters taken at $r_0 = \sqrt{x_0^2 + y_0^2}$. Equation (31) has been derived under the assumption $\dot{y} = 0$ at the injection point. It can be verified that at infinity ($r_0 \rightarrow \infty$) b approaches y_0 as it should according to the definition of the impact parameter. In anisotropic spacetimes, the injection momentum $\mathbf{p} = p_x \hat{\mathbf{x}} + p_y \hat{\mathbf{y}}$ is not collinear with the propagation direction. Since we impose that initially the ray propagates along the x -axis, at the injection point the momentum forms an angle given by

$$\frac{p_{y0}}{p_{x0}} = \frac{(1-f)x_0 y_0}{x_0^2 + f y_0^2} = \tan \alpha. \quad (32)$$

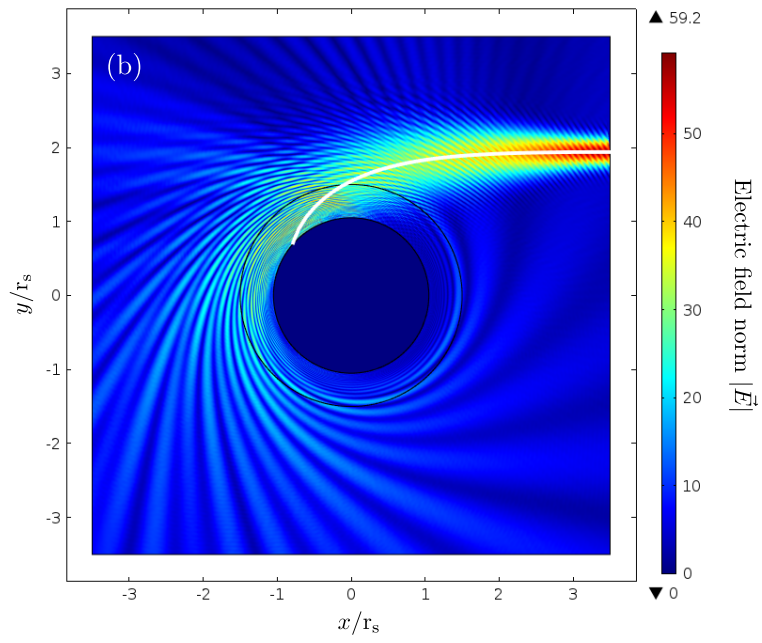
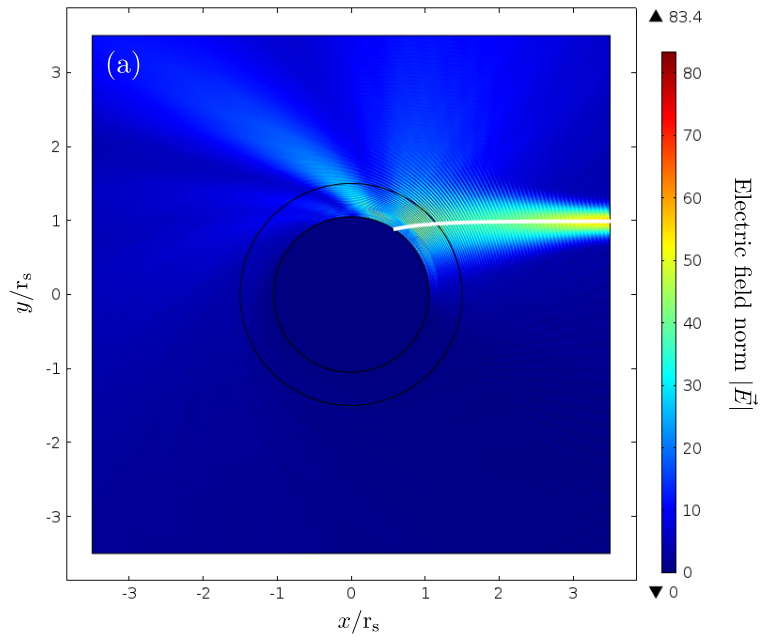
It is exactly the same angle α between the wave normal and the Poynting vector and between the \mathbf{B} and \mathbf{H} fields, all given by (11). Here we clearly see how the spacetime anisotropy results in the anisotropy of the medium.

In Fig. 2 we show the results of numerical simulations for Gaussian beams incident with different impact parameters b for the case of a Schwarzschild anisotropic spacetime. The ray paths calculated from the Hamilton equations are superimposed in white on the wave pattern (shown in color) simulated by solving the wave equation. We observe a good correspondence between the two approaches: the ray path follows the center of the Gaussian beam even when it deflects. Moreover, the ray trajectories are consistent with the behavior expected from theory: we observe the ray capture for $b < b_c$ [Figs. 2(a),(b)], and its deflection for $b > b_c$ [Figs. 2(c),(d)]. The wave solutions present a more complex behavior due to interference effects. The Gaussian beam splits into a set of rays or “subbeams”, one part of it is captured due to having $b < b_c$ and another part bends around the photonic sphere of the black hole and interferes with the primary beam. This effect is more pronounced in Figs. 2(b),(c) for the values of b which are not very far from the critical value $b_c \approx 2.598r_s$. It is interesting to see that the subbeams are regularly spaced while bending around

the black hole and some of them exhibit smaller subwavelength ripples. Similar behavior was observed in Ref. [19] for a larger value of the wavelength.

In Fig. 3 we present the results for the isotropic case of the Schwarzschild metric using the same set of the impact parameters b as for the anisotropic case of Fig. 2. Again, we observe a good agreement between the wave propagation and the ray tracing. The principal difference is that now the local velocity of light becomes isotropic and it is determined by a scalar function – the refractive index $n(\rho)$. As a consequence, the interference pattern we observe is substantially different. One can see irregular “jets” leaving the system in radial directions, similar to those obtained by Genov et al. for the isotropic spacetimes [14]. The small ripples are also clearly seen. Since we simulate an isotropic medium, another distinctive feature we observe, is that the wavefronts are perpendicular to the beam propagation, i.e. $\alpha = 0$.

On the other hand, if we compare the ray trajectories for the isotropic and the anisotropic cases, they are rather similar: the same kinds of orbits (capture or deflection) are observed corresponding to the same value of the impact parameter b . Notice that the parameter b is a coordinate-independent quantity, since it is defined as a ratio between the angular momentum and energy of the photon measured at infinity. Other physical properties, like the deflection angle between the incoming and outgoing light rays measured by a distant observer at infinity should also be equal independently of the coordinate system used.



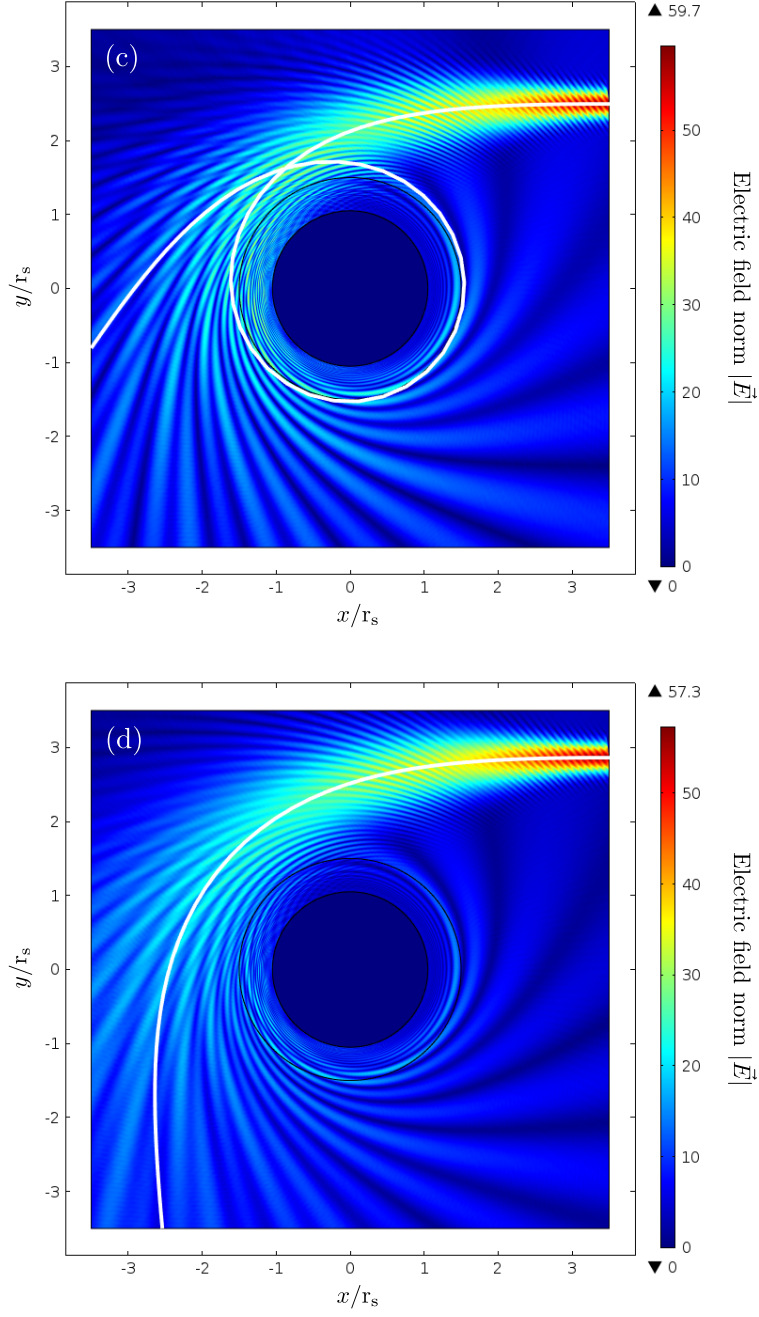
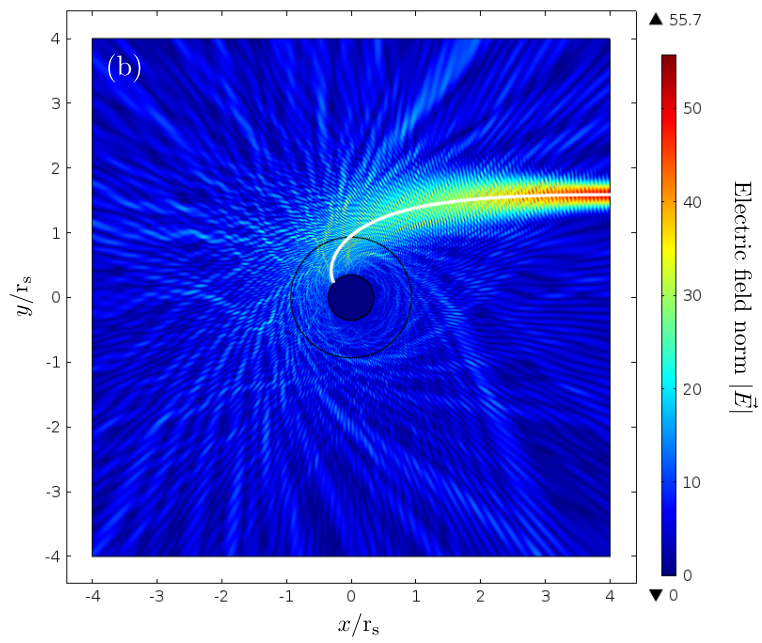
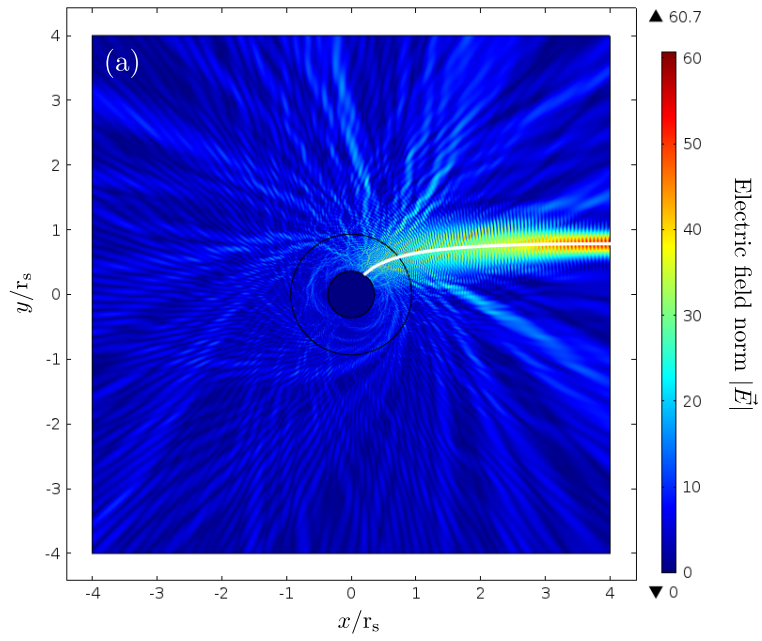


Figure 2: TE Gaussian beam of wavelength $\lambda = 0.15r_s$ compared with ray path (superimposed in white) in metamaterial mimicking the Schwarzschild anisotropic spacetime for different impact parameters: (a) $b = r_s$, (b) $b = 2r_s$, (c) $b \approx b_c$ (slightly above), and (d) $b = 3r_s$. The event horizon r_s and the photon sphere of radius $3r_s/2$ are depicted by black circles.



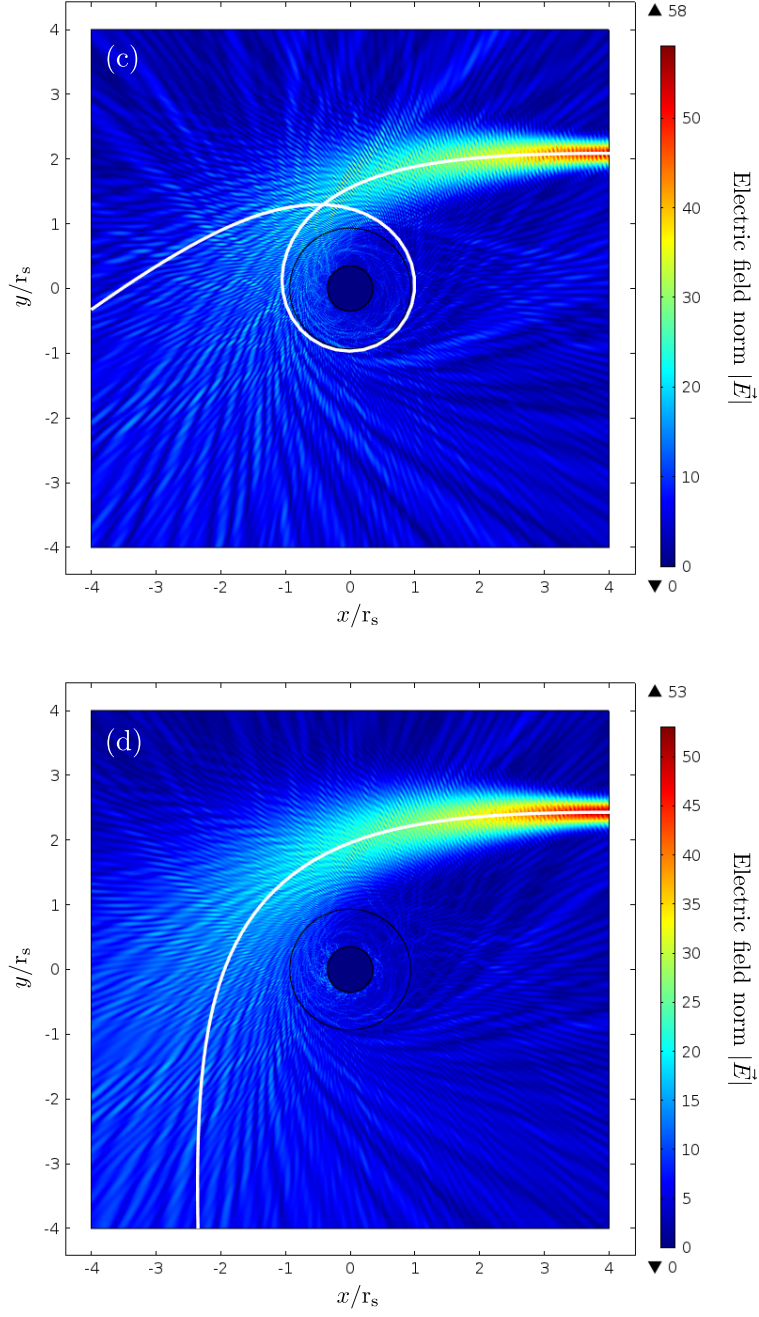


Figure 3: TE Gaussian beam of wavelength $\lambda = 0.15r_s$ compared with ray path (superimposed in white) in metamaterial mimicking the Schwarzschild isotropic spacetime for different impact parameters: (a) $b = r_s$, (b) $b = 2r_s$, (c) $b \approx b_c$ (slightly above), and (d) $b = 3r_s$. The event horizon ρ_s and the photon sphere of radius $(2 + \sqrt{3})r_s/4$ are depicted by black circles.

6. Concluding remarks

In conclusion, we have derived the constitutive relations of an inhomogeneous anisotropic medium which is formally equivalent to a class of static spacetime metrics associated with a spherically symmetric cosmological black hole. It is known that every spacetime metric can be written in different coordinate systems and therefore, can be projected in many different ways into the corresponding effective medium, each with specific material properties. This procedure is at the basis of the transformation optics [3, 4]. In particular, a static spherically symmetric spacetime written in arbitrary (non-isotropic) coordinates can be equivalently described in isotropic coordinates (conformally flat form) after performing the appropriate coordinate transformation.

We have analyzed the medium properties and the propagation of light through the effective media corresponding to these two cases of interest: anisotropic and isotropic. For the anisotropic spacetime, we found that only one kind of the medium anisotropy would be essential: either magnetic or electric, for TE or TM polarized waves respectively. In both cases, light does not propagate in the direction of the wave normal. There appears an angle between the wave velocity and the ray velocity, specified by Eq. (11), which is related to the anisotropic factor of the metric. For some spacetimes, this angle can even be negative giving such interesting phenomenon as negatively refracting medium [37]. However, this cannot be observed for the metric (5).

It is interesting to note that for the kind of metric we consider, the time and space metric coefficients correspond to different properties of the medium: either electric or magnetic. For instance, for the TE wave, the time dilation term determines the dielectric permittivity of the equivalent medium, while the spatial metric components correspond to its magnetic permeability. This phenomenon cannot be seen in the isotropic medium characterized by the scalar refractive index. We presented the results for the TE wave in the magnetically anisotropic media, although the case of the TM wave in the electrically anisotropic media perhaps would be easier to realize in the laboratory experiment. Both considerations are theoretically equivalent.

By applying the eikonal approximation to the wave equation, we have also obtained the expression for the optical Hamiltonian which was used to simulate the ray paths. It coincides with the Hamiltonian obtained from general relativity for null-geodesics. It is clearly seen from our theoretical approach, that despite the fact that in the geometrical optics limit different media may give the same ray paths, their full-wave description will differ. Taking as an example the Schwarzschild spacetime metric, we obtained a very good correspondence between the wave propagation and the ray trajectories for the anisotropic and isotropic media. Although the isotropic case is easier to implement in metamaterials, we believe that due to a rapid progress in metamaterial technology [38] (see also proposal [22]), anisotropic media will also be available to model the cosmological phenomena in the laboratory and many physically interesting phenomena could be observed.

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